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Quantum Hall wave functions based on S_3 conformal field theories

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We construct a family of quantum Hall Hamiltonians whose ground states, at least for small system sizes, give correlators of the S_3 conformal field theories. The ground states are considered as trial wave functions for quantum Hall effect of bosons at filling fraction $\nu=3/4$ interacting either via delta function interaction or delta function plus dipole interaction. While the S_3 theories can be either unitary or nonunitary, we find high overlaps with exact diagonalizations only for the nonunitary case, suggesting that these wave functions may correspond to critical points, possibly analogous to the previously studied Gaffnian wave function. These wave functions give an explicit example which cannot be fully characterized by their thin-torus limit or by their pattern of zeros.

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One of the major breakthroughs in the theory of quantum Hall effect was the realization of the close correspondence between quantum Hall wave functions and conformal field theories (CFTs). This correspondence suggested the possibility that quasiparticle excitations of certain quantum Hall states might have nontrivial ("non-Abelian") braiding statistics[1]—a property that, if true, could be useful for error resistant quantum information processing.[2] While this CFT correspondence has been extremely powerful, only a very few nontrivial CFTs have successfully been used to generate reasonable trial wave functions[1,3,4] (we define "success" in a moment). In fact, among spin-polarized single-component wave functions (which we will focus on throughout this Rapid Communication[4]), it appears that the only successes of this approach have been the Read-Rezayi[3] series including the Moore-Read[1] state and the Laughlin state. While many other wave functions have been proposed,[5–10] serious problems plague these attempts: (1) CFT approaches that do not produce an explicit wave function[5,6] are difficult to study. (2) Of the new explicit wave functions that have been proposed, many do not correspond to rational unitary CFTs,[7–10] and there is increasingly strong evidence[11] that only rational unitary CFTs can describe a gapped phase of matter (although other CFTs may describe interesting critical points between gapped phases, and may therefore be worth studying, nonetheless). (3) With the exception of the above-mentioned successes, the nonunitary Gaffnian[7] and the nonrational Haffnian,[8] no one has found an explicit Hamiltonian whose ground state is one of these proposed CFTs. (4) Even if one proposes a Hamiltonian that produces a valid unitary CFT wave function as its ground state, there is still a serious issue of whether this phase of matter can be realized in any reasonable experiment. We define success of a wave function by these four criteria. In passing, we note two other partially successful wave-function constructions in Refs. 12 and 13 which both fail condition (3).

In the current Rapid Communication we propose a family of wave functions based on the so-called S_3 CFTs,[14] which describe bosons at $\nu=3/4$ (or fermions at $\nu=3/7$). While far from declaring these wave functions to be "successful" on the scale of the Read-Rezayi series,[3] our results are nonetheless favorable with respect to the above listed criteria. In particular, we develop a family of Hamiltonians that, at least for small systems, generate a family of S_3 CFT wave functions, which includes both unitary and nonunitary cases. We find that these trial wave functions can have very high overlap with wave functions of exact diagonalizations of potentially realistic Hamiltonians corresponding to rotating Bose gases.[15] Interestingly, we find that the high overlaps coincide with nonunitary S_3 wave functions. This behavior, reminiscent of the previously studied Gaffnian wave function,[7] suggests criticality in both cases. We believe that these results may shed some important light on the general applicability of nonunitary CFTs to quantum Hall physics in general. Further, our results suggest that wave functions based on the higher generation parafermion CFTs (Refs. 6, 10, and 16) may generally be of experimental and theoretical interest.

The S_3 wave functions are within the larger class of generalized parafermion (or W-algebra) wave functions[6,10,16] that generalize the parafermionic Read-Rezayi states.[3] It is useful to start by reviewing some of the properties of the S_3 CFTs (Ref. 14) before describing the wave functions that can be built from them.

The family of S_3 CFTs have two simple currents ψ and $\psi^\dagger$ with conformal dimension $h=4/3$ (these are analogs of ψ_1 and ψ_2 of the $\mathbb{Z}_3$ parafermion theory[14,16]). These fields satisfy the operator product expansions (OPEs)

$$\psi(z_1)\psi(z_2) = \lambda(z_1-z_2)^{-4/3}\psi^\dagger(z_2) + \cdots, \tag{1}$$

$$\psi^\dagger(z_1)\psi^\dagger(z_2) = \lambda(z_1-z_2)^{-4/3}\psi(z_2) + \cdots, \tag{2}$$

$$\psi(z_1)\psi^\dagger(z_2) = (z_1-z_2)^{-8/3}I(z_2) + \cdots, \tag{3}$$

where I is the identity field, c is the central charge, and $\cdots$ represents terms less singular by integer powers of (z_1-z_2). The constant λ is related to the central charge by[14] $\lambda^2=4(8-c)/(9c)$. Within this family of CFTs there exists a series of rational minimal models, which we denote $S_3(p,p')$ having corresponding central charge $c=2\{1-[3(p-p')^2/4pp']\}$ for p and p' positive integers. Note that there is a rational minimal CFT from this family arbitrarily close to any $c\leq 2$, al-

TABLE I. Examples of S_3 wave functions; c is the central charge and θ is the tuning parameter of the Hamiltonian. Values of θ/π are approximate (except 0 and 0.5) and are calculated for the $N=12$ system size. The $\mathbb{Z}_6$ case condenses the ψ_2 field as discussed originally in Ref. 3 and developed further in Ref. 22. The Jack wave function is the $(k,r)=(3,4)$ case from Ref. 9. The $c=8$ case is the $\nu=3/4$ wave function from Ref. 7, and $c=-4$ is from Ref. 21. The accumulation point with $c=2$ is the symmetrized product of three $\nu=1/4$ Laughlin wave functions.

Unitary theories		c	θ/π
$S_3(3,7)$	Tricritical Potts	6/7	0.288
$S_3(4,8)$	$\mathbb{Z}_6$ parafermion	5/4	0.378
$S_3(5,9)$		22/15	0.420
$S_3(6,10)$	$[\mathbb{Z}_3$ parafermion$]^2$	8/5	0.443
	⋮	⋮	⋮
	Accumulation point	2	0.5
Nonunitary rational examples		c	θ/π
$S_3(2,7)$		−19/28	−0.0113
$S_3(3,10)$		−9/20	0.0176
$S_3(1,7)$	Jack	−40/7	−0.188
$S_3(1,3)$		0	0.0913
Other Examples		c	θ/π
		8	−0.307
		−4	−0.166
		−1	−0.0443
		−0.584…	0

though the only unitary members of this set occur for the discrete series $p'=p+4\geq 7$. See Table I for several examples of such minimal models.

We will focus on the case of quantum Hall effect of bosons, although the fermionic wave functions can also be considered quite analogously (as we will see below, the bosonic case seems to be of potential experimental interest). Following the general approach for constructing quantum Hall wave functions from Refs. 1 and 3, we write a wave function for bosons at filling fraction $\nu=3/4$ as

$$\Psi=\langle\psi(z_1)\cdots\psi(z_N)\rangle\prod_{i<j}(z_i-z_j)^{4/3}, \tag{4}$$

where the number of particles N is taken to be divisible by 3. The full wave function also includes a measure $\prod_{i=1}^{N}\mu_i$, which we do not write explicitly. For a planar geometry the measure is $\mu_i=\exp(-|z_i|^2/4\ell_0^2)$ with ℓ_0 as the magnetic length, whereas for a spherical geometry[3] $\mu_i=(1+|z_i|^2/4R^2)^{-(1+N_\phi/2)}$, where R is the radius of the sphere and N_ϕ is the total number of flux quanta through the sphere. Note that on the sphere this wave function occurs for flux related to the number of particles by $N_\phi=\frac{4}{3}N-4$ (i.e., the shift is 4).

From the OPEs it is easy to see that Ψ does not vanish when three particles come to the same position, but vanishes as four powers as the fourth particle arrives at this position. (This generalizes the $\mathbb{Z}_3$ Read-Rezayi state which vanishes as two powers as the fourth particle arrives.) As the four particles come together, the wave function must vanish proportional to some fourth degree translationally invariant symmetric polynomial. As pointed out in Ref. 17, there is a two-dimensional space of such polynomials. Let us parametrize this with orthonormal vectors

$$|\varphi_\theta^0\rangle=\cos\theta\,|\varphi_1\rangle+\sin\theta\,|\varphi_2\rangle, \tag{5}$$

$$|\varphi_\theta^\perp\rangle=-\sin\theta\,|\varphi_1\rangle+\cos\theta\,|\varphi_2\rangle, \tag{6}$$

where $\varphi_1=c_1\Sigma_{1\leq i<j\leq 4}(z_i-z_j)^4$ and φ_2 is chosen orthogonal to φ_1 with respect to the measure μ_i. On the plane, $\varphi_2=c_2(z_1+z_2+z_3-3z_4)\times$cyclic, where c_1 and c_2 are normalizations, such that $\langle\varphi_i|\varphi_j\rangle=\delta_{ij}$ for $i=1,2$ so that $\langle\varphi_\theta^k|\varphi_\theta^m\rangle=\delta_{km}$, where $k,m=0,\perp$.

We now define a Hamiltonian as outlined in Ref. 17, $H=\tilde{H}+H_\theta$, where $\tilde{H}$ gives positive energy to any four particles having a relative angular momentum of less than 4, and

$$H_\theta=\sum_{i,j,k,l}|\varphi_\theta^0(z_i,z_j,z_k,z_l)\rangle\langle\varphi_\theta^0(z_i,z_j,z_k,z_l)| \tag{7}$$

forces any four particles to approach each other proportional to $\varphi_\theta^\perp$ (or as a higher degree polynomial) or else they will pay an energy penalty. Thus, this Hamiltonian is projecting the four-particle behavior to be $\varphi_\theta^\perp$. Using the approach of Ref. 17 this Hamiltonian may be written either in terms of four-particle pseudopotentials or in terms of derivatives of a four-particle delta function. While we have no proof that our Hamiltonian will result in a unique ground state generally, we find numerically that for up to 15 bosons on a sphere, the ground state (for $N_\phi=\frac{4}{3}N-4$ with N divisible by 3) is indeed unique, and therefore must correspond to the CFT generated wave function (4) for the appropriate central charge corresponding to the chosen θ. (Furthermore, it is found that the ground-state wave functions generally satisfy the product rule discovered in Ref. 18.) To identify the central charge associated with a given H_θ consider the limits

$$G=\lim_{z\to 0}(z^{-4}[\lim_{z_4\to z_3=z}[\lim_{z_2\to z_1=0}\Psi]]), \tag{8}$$

$$F=\lim_{z_4\to z\to 0}(z^{-4}[\lim_{z_3\to 0}[\lim_{z_2\to z_1=0}\Psi]]). \tag{9}$$

From the OPEs it is easy to show that

$$G/F=\lambda^2=4(8-c)/(9c). \tag{10}$$

Thus, taking the analogous limits for the four-particle wave function $\varphi_\theta^\perp$ (which gives the limiting four-particle behavior of the ground state), one can easily determine c for any given θ. We are thus able to numerically generate the S_3 quantum Hall wave functions [Eq. (4)] corresponding to any central charge. For certain values of the central charge, the generated wave function results in wave functions previously proposed (see Table I). Note that generating the ground-state wave function does not address a host of questions, such as whether the same Hamiltonian will produce a unique ground state for $N>15$ (although this seems likely), whether this

Hamiltonian is gapped, what the excitations look like, or what the spectrum is when additional flux quanta are added.[19]

The generated wave functions, since they do not vanish when three particles come to the same point but vanish as four powers when the fourth arrives, are examples of cluster state wave functions as discussed, for example, in Refs. 9, 20, 22, and 23. However, we emphasize again that this statement does not by itself fully determine the wave function—one needs to specify $\varphi_\theta^\perp$, the precise manner in which the wave function vanishes. This means that neither the thin-torus limit[20] (or root state[9] of the full clustered polynomial fractional quantum Hall wave function[9,23]) nor the pattern of zeros[22] uniquely specifies the wave function. Thus, our current example starkly points out the weaknesses of several proposed schemes for characterizing quantum Hall wave functions in general.

We now turn to the question of whether these wave functions could have applicability to experimentally realistic situations. We choose to look at quantum Hall effect of bosons, which may be relevant to rotating cold Bose gases.[15] Assuming delta function interactions, Ref. 24 showed compellingly that both filling fractions $\nu=1/2$ (where the ground state is exactly the Laughlin wave function) and $\nu=2/3$ are well described by composite fermion physics (meaning both the ground and excited states are well described). However, the next member of the composite fermion series, $\nu=3/4$, while still a gapped state,[25] fits much less well to the composite fermion description.[24] This was one of the reasons why we began to seek an alternate wave function for this state. Note that the composite fermion wave function and our S_3 wave functions compete with each other directly being that on the sphere they both occur for $N_\phi=\frac{4}{3}N-4$. Within this work we also consider altering the delta function interaction by adding an additional dipolar interaction as well,[15,26] which also could be experimentally relevant. We quantify the amount of dipole interaction we have added by specifying the so-called Haldane pseudopotential coefficient ratio V_2/V_0.

In Fig. 1, we show the overlap of our S_3 trial wave functions with the exact ground state for $N=12$ bosons at $N_\phi=\frac{4}{3}N-4=12$, as a function of the tuning parameter θ in the Hamiltonian. The solid line is the overlap with the exact ground state of bosons interacting via delta function interactions. The $L=0$ Hilbert space of this system has 127 dimensions, so the high overlaps of over 85% should be considered to be significant. When we include dipolar interaction terms as well, the overlap increases further, as shown in the inset, reaching over 94% near $V_2/V_0=0.2$. This increase in the overlap with the added dipole interaction is reminiscent of the behavior of bosons at $\nu=1$ where,[27] for similar sized systems, the overlap of the Moore-Read wave function with the exact ground state at $V_2/V_0=0$ is about 88%, increases to about 95% with increasing V_2/V_0, and then collapses above $V_2/V_0\approx 0.2$. In comparison, the overlap here of the composite fermion wave function with the exact ground state[24] is only about 74% at $V_2/V_0=0$, then decreases monotonically with increasing V_2/V_0. The dashed line in the main plot shows the overlap of our trial wave functions with the exact ground state of bosons interacting via delta function plus dipole interaction with $V_2/V_0=0.2$.

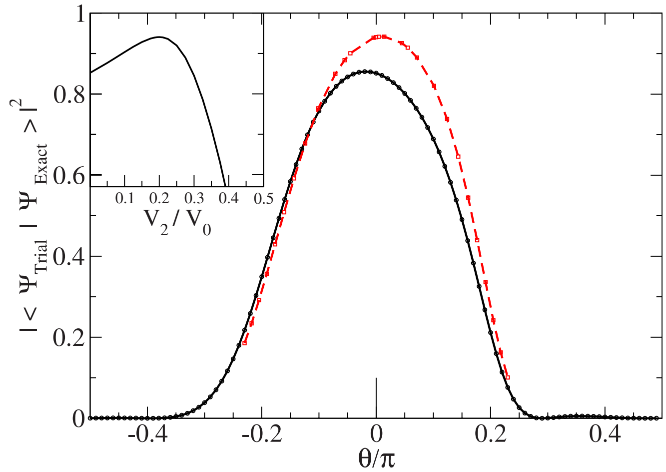

FIG. 1. (Color online) Squared overlap of $N=12$ bosons at $\nu=3/4$ exact wave function with the S_3 trial wave function as a function of the tuning parameter θ. Solid curve shows the overlap of the trial wave functions with the exact ground state of bosons interacting via repulsive delta function interaction. Dashed curve is the overlap with the exact ground state of bosons interacting via repulsive delta function plus dipole interaction where the amount of dipole interaction is set such that $V_2/V_0=0.2$. Inset: overlap at $\theta=0$ as a function of amount of dipole interaction V_2/V_0. The vertical axis of the inset is aligned with the vertical axis of the main plot.

The maximum overlaps in these diagonalizations occur near $\theta=0$, which corresponds to a wave function that vanishes proportional to $\varphi_0^\perp=\varphi_2$ when four particles come together. For $N=12$ as shown in the figure, $\theta=0$ corresponds to central charge $c\approx -0.584$. (In the thermodynamic limit $\theta=0$ corresponds to $c=-8/11$.) Numerically, the maximum overlap with the ground state of delta function interaction (solid line) occurs at $c\approx -0.739$ whereas for the case of delta function plus dipole with $V_2/V_0=0.2$ (dashed plot) the maximum occurs for $c\approx -0.503$. At any rate, these negative values of the central charge indicate that such a CFT must necessarily be nonunitary. The entire unitary series occurs for $0.25<\theta/\pi\leq 0.5$ and the overlaps with the exact ground states are low. In searching for a CFT that corresponds to our data, it is worth noting that we can always find a rational (but not typically unitary) CFT arbitrarily close to any desired central charge with $c<2$ by choosing appropriate p and p' in $S_3(p,p')$. However, if we insist that p and p' are both "reasonably small" (say 10 or less), then the only theories that give $-1<c<0$ are $S_3(2,7)$ and $S_3(3,10)$ (see Table I).

Our results are reminiscent of the physics of the Gaffnian[7] ($\nu=2/3$ for bosons) in many ways. In both cases we find very high overlaps with exact diagonalization, despite being nonunitary theories. In both cases, there is a competing composite fermion wave function at the same flux which also has high overlap with the exact ground state. Our S_3 wave function vanishes roughly proportional to $\varphi_2=[(z_1+z_2+z_3-3z_4)\times\text{cyclic}]$ when four particles come together, whereas the Gaffnian vanishes analogously as $[(z_1+z_2-2z_3)\times\text{cyclic}]$ when three particles come together. This analogy suggests that the entire composite fermion series will have similar behavior and that the kth member of the composite fermion series [$\nu=k/(k+1)$ for bosons or $k/(2k+1)$ for fermions] will compete similarly with a generalized parafermion wave function[6,10,16] with $\mathbb{Z}_k$ symmetry.

Our current understanding of the Gaffnian suggests that,[7] as a nonunitary CFT, it is actually a critical point between two phases—one of which is the composite fermion phase and the other (less well understood) phase perhaps is some sort of a strong pairing phase. One can explicitly tune through this critical point by varying the two-particle V_0 interaction: for positive V_0 the wave function has increasingly high overlap with the composite fermion wave functions, whereas for negative values of V_0 the wave function rapidly obtains very low overlaps. A calculation of wave-function fidelity (to be published) suggests that the critical point is at $V_0=0$. Further, one may conjecture that the high overlaps between the Gaffnian and the composite fermion wave function at the same filling fraction are a sign that the typical composite fermion wave functions are somehow "close" to this critical point. We conjecture that there may be similar physics occurring for this case of $\nu=3/4$ bosons (or $\nu=3/7$ fermions) where the critical theory here is one of the S_3 wave functions. Indeed, the behavior with an added V_0 interaction appears to mimic that of the Gaffnian quite closely: positive V_0 leaves the wave function relatively stable with high composite fermion overlap, whereas negative V_0 pushes the wave function to a different phase which has very low overlap with composite fermions. We conjecture that the entire composite fermion series follows this pattern.

As a summary, let us now revisit our above criteria for a successful wave function. (1) While we have not written an analytical wave function, we have nonetheless numerically generated the wave function corresponding to the S_3 CFTs at least for small systems. (2) At least some members of this family are unitary CFTs. (3) We have written explicit Hamiltonians for which these wave functions are the unique ground states, at least for small systems ($N \leq 15$). (4) Certain wave functions from this family, albeit the nonunitary members of the family, have very high overlap with the ground state of potentially experimentally relevant Hamiltonians.

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